Can a coherent risk measure be too subadditive?

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Abstract

We consider the problem of determining appropriate solvency capital requirements for an insurance company or a financial institution. We demonstrate that the subadditivity condition that is often imposed on solvency capital principles can lead to the undesirable situation where the shortfall risk increases by a merger. We propose to complement the subadditivity condition by a *regulator's condition*. We find that for an explicitly specified confidence level, the Value-at-Risk satisfies the regulator's condition and is the "most efficient" capital requirement in the sense that it minimizes some reasonable cost function. Within the class of concave distortion risk measures, of which the elements, in contrast to the Value-at-Risk, exhibit the subadditivity property, we find that, again for an explicitly specified confidence level, the Tail-Value-at-Risk is the optimal capital requirement satisfying the regulator's condition.

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1 Introduction

In a perfect capital market, due to the Modigliani-Miller irrelevance theorems, insurance companies should not be concerned with risk management and capital allocation. In reality, besides taxes, a main capital market imperfection is asymmetric information, which makes it expensive for insurers to attract external funds, to add to existing internal capital. Even without being in financial distress, which would further increase the cost of external funding, insurers can lose the opportunity of pursuing profitable new investments when internal capital has been depleted and external capital is available only at high costs. This provides an incentive for the shareholders and management of an insurance company to avoid holding too little capital. At the same time, it is clear that holding too much capital is costly.

As far as the policyholders is concerned, there is considerable empirical evidence that the demand for insurance is sensitive to the solvency position of the insurer; see Froot (2005) and the references therein. The legislative power with respect to the protection of policyholders against insolvency, is usually assigned to a regulatory authority, the supervisor. Our main concern in this paper will be the point of view of the regulator. The regulatory authority monitors the solvency position of the insurers in order to protect the contingent claims of the policyholders.

In an insurance business the production cycle is inverted, because premiums are paid by the policyholder before claims are paid by the insurer. An insurance portfolio may get into distress when it turns out that claims exceed the corresponding premiums, as in that case the obligations towards the insureds cannot be completely covered. In order to protect the policyholders, the regulatory authority in force will impose a *solvency capital requirement*. This means that the regulator requires the available capital that the company holds, which is the surplus of assets over liabilities, to be of some minimal level, depending on the riskiness of the business under consideration. This capital serves as a buffer against the risk that premiums will turn out to be insufficient to cover future policyholder claims. Although in principle the regulator wants the solvency capital requirement as large as possible, there clearly is a limitation on the capital cost burden that it can impose on the insurer.

In this paper we investigate the use of risk measures for setting solvency capital requirements. We demonstrate that *coherent* risk measures (as defined by Artzner *et al.* (1999)) used as solvency capital requirements can be too subadditive, in the sense that they may lead to an increase of the shortfall risk in case of a merger, a property that will be undesirable from the regulator's point of view. We propose to complement the subadditivity condition by a *regulator's condition*. We find that for an explicitly specified confidence level, the Value-at-Risk satisfies the regulator's condition and is the "most efficient" capital requirement in the sense that it minimizes some reasonable cost function. Within the class of concave distortion risk measures, of which the elements, in contrast to the Value-at-Risk, exhibit the subadditivity property, we find that, again for an explicitly specified confidence level, the Tail-Value-at-Risk is the optimal capital requirement satisfying the regulator's condition. Although we will primarily focus on solvency capital requirements for an insurance portfolio, the results presented hold more generally for any (re)insurance company or financial institution supervised by a regulatory authority.

We have chosen to use the general term "risk measure" in the title of this paper, since the use of this term is widespread in the literature, often without specifying the particular context and characteristics of the risk to be measured. But context matters. We emphasize that, instead of the term "risk measure", it might be more appropriate here to use the more explicit term "solvency capital requirement".

This paper is organized as follows: in Section 2, we introduce (classes of) risk measures and discuss some of their properties. In Section 3, we propose a method to determine solvency capital requirements as the minimum of a cost function taking into account the shortfall risk and the cost of capital. Section 4 discusses the subadditivity property. In Section 5, we investigate the problem of avoiding that a merger increases the shortfall risk. A new axiom, which we call the "regulator's condition" and which can be used to complement the subadditivity axiom, is introduced in Section 6. Finally, Section 7 concludes the paper.

2 Risk measures

Consider a set Γ of real-valued random variables defined on a given measurable space $(\Omega, \mathcal{F})$. We will assume that $X_1, X_2 \in \Gamma$ implies that $X_1 + X_2 \in \Gamma$, and also $aX_1 \in \Gamma$ for any $a > 0$ and $X_1 + b \in \Gamma$ for any real b. A functional $\rho : \Gamma \to \mathbb{R}$, assigning a real number to every element of Γ, is called a *risk measure* (with domain Γ).

In the sequel, we will interpret Ω as the set of states of nature at the end of some fixed reference period, for instance one year. The set Γ will be interpreted as the extended set of losses at the end of the reference period, related to insurance portfolios that a particular regulatory authority controls.

Let X be an element of Γ. In case all claims of the corresponding insurance portfolio are settled at the end of the reference period and all premiums are paid at the beginning of the reference period, the (aggregate) loss X can be defined as claims minus the sum of premiums and investment income. In a more general setting, we can define X as the sum of the claims to be paid out over the reference period and the provisions to be set up at the end of the reference period, minus the sum of the provisions available at the beginning of the reference period, the investment income and the premiums received over the reference period. Here, claims, premiums and provisions are understood as gross amounts, i.e., including expenses. The valuation principles on the basis of which the value of the assets (represented by the provisions available, the premiums received and the investment income generated) and in particular the liabilities (represented by the provisions to be set up and the claims to be paid out), are left unspecified in this paper; our setup is compatible with any particular valuation basis.

A portfolio faces insolvency in case its loss X is positive. In this case the obligations towards the policyholders cannot be completely covered. Solvency reflects the financial capacity of a particular risky business to meet its contractual obligations. To protect the policyholders from insolvency, the regulatory authority imposes a *solvency capital requirement* $\rho[X]$, which means that the available capital in the company has to be at least equal to $\rho[X]$. This capital can be employed when premiums and provisions together with the investment income, turn out to be insufficient to cover the policyholders' claims. In principle, $\rho[X]$ will be chosen such that one can be "fairly sure" that the event "$X > \rho[X]$" will not occur.

Although we will stick to the interpretation of loss as introduced above, most of the results in this paper also hold for other interpretations of the elements of Γ. In case of a retail bank for instance, one can define X as the future difference between the value of the liabilities (in this case mostly savings accounts) and the value of the assets (typically loans and mortgages). The value of the bank's assets is subject to changes in interest rates, credit spreads and the occurrence of defaults during the reference period. The value of the bank's liabilities also depends on the level of interest rates, but is furthermore subject to, for example, operational risk that the bank faces.

We fix a *base probability measure* P on $\mathcal{F}$. The base probability measure could be the "physical probability measure", but could also be another (for example, subjective or risk-neutral) probability measure. Two well-known risk measures used for setting solvency capital requirements are the *Value-at-Risk* and the *Tail-Value-at-Risk*.[1] For a given probability level p they are denoted by Q_p and TVaR_p, respectively. They are defined by

$$Q_p[X] = \inf\{x \mid P[X \leq x] \geq p\}, \qquad 0 < p < 1, \tag{1}$$

[1]Between these two, the Value-at-Risk is currently by far the most popular risk measure in practice, among both regulators and risk managers; see, for example, Jorion (2001).

and

$$\mathrm{TVaR}_p[X] = \frac{1}{1-p}\int_p^1 Q_q[X]\ dq, \qquad 0 < p < 1. \tag{2}$$

The *shortfall* of the portfolio with loss X and solvency capital requirement $\rho[X]$ is defined by

$$\max(0, X - \rho[X]) \equiv (X - \rho[X])_+ . \tag{3}$$

The shortfall can be interpreted as that part of the loss that cannot be covered by the insurer. It could also be referred to as the *residual risk*, the *insolvency risk* or the *policy-holders' deficit*.

As is well-known, $\mathrm{TVaR}_p[X]$ can be expressed as a linear combination of the corresponding quantile and its expected shortfall:

$$\mathrm{TVaR}_p[X] = Q_p[X] + \frac{1}{1-p}E\left[(X - Q_p[X])_+\right], \tag{4}$$

where the expectation is taken with respect to the base probability measure P.

Properties of risk measures have been investigated extensively; see e.g., Goovaerts, De Vylder & Haezendonck (1984). Some well-known properties that risk measures may (or may not) satisfy are monotonicity, positive homogeneity, translation invariance, subadditivity, convexity and additivity for comonotonic risks. They are defined as follows:

- *Monotonicity*: for any $X_1, X_2 \in \Gamma$, $X_1 \leq X_2$ implies $\rho[X_1] \leq \rho[X_2]$.
- *Positive homogeneity*: for any $X \in \Gamma$ and $a > 0$, $\rho[aX] = a\rho[X]$.
- *Translation invariance*: for any $X \in \Gamma$ and $b \in \mathbb{R}$, $\rho[X + b] = \rho[X] + b$.
- *Subadditivity*: for any $X_1, X_2 \in \Gamma$, $\rho[X_1 + X_2] \leq \rho[X_1] + \rho[X_2]$.
- *Convexity*: for any $X_1, X_2 \in \Gamma$ and $\lambda \in [0, 1]$, $\rho[\lambda X_1 + (1-\lambda) X_2] \leq \lambda\ \rho[X_1] + (1-\lambda)\ \rho[X_2]$.
- *Comonotonic additivity*: for any $X_1, X_2 \in \Gamma$ that are comonotonic, $\rho[X_1 + X_2] = \rho[X_1] + \rho[X_2]$.

Here, and in the remainder of this paper, a stochastic inequality $X_1 \leq X_2$ has to be understood as $X_1(\omega) \leq X_2(\omega)$ for all $\omega \in \Omega$. Such inequality implies a (P-)almost sure inequality for any probability measure on the measurable space. We recall that the random couple (X_1, X_2) is said to be *comonotonic* if there is no pair $\omega_1, \omega_2 \in \Omega$ such that $X_1(\omega_1) < X_1(\omega_2)$ while $X_2(\omega_1) > X_2(\omega_2)$; see Denneberg (1994). Equivalently, comonotonic random variables can be characterized as being non-decreasing functions of a common random variable. Comonotonicity is a very strong positive dependence notion and essentially reduces multivariate randomness to univariate randomness. Theoretical and practical aspects of the concept of comonotonicity in insurance and finance are considered in Dhaene *et al.* (2002a,b).

In the sequel, when we consider losses X_j, we always assume that they are elements of Γ. Also, when we mention that a risk measure satisfies a certain property, it has to be interpreted as that it satisfies this property on Γ.

The desirability of the subadditivity property of risk measures has been a major topic for research and discussion; see also Section 4 of this paper. As is well-known, the Value-at-Risk does not in general satisfy the subadditivity property (although it does in various particular cases), whereas for any p the Tail-Value-at-Risk does.

In Artzner *et al.* (1999), a risk measure that satisfies the properties of monotonicity, positive homogeneity, translation invariance and (most noticeably) subadditivity is called a *coherent* risk measure. Huber (1981), in a different context, defines the *upper expectation* ρ_Π, induced by a subset Π of the set of all probability measures on the measurable space $(\Omega, \mathcal{F})$, as the risk measure that attaches to any loss X the real number $\rho_\Pi[X]$ given by

$$\rho_\Pi[X] = \sup\{E_P[X] \mid P \in \Pi\}. \tag{5}$$

Huber (1981) proves for the case of a finite set Ω, that a risk measure satisfies monotonicity, positive homogeneity, translation invariance and subadditivity (and hence is coherent as defined by Artzner *et al.* (1999)) if and only if it has an upper expectation representation. This result remains valid in more general spaces (see Delbaen (2002) for details). Artzner *et al.* (1999) call the elements of Π *generalized scenarios*.

Wang (1996) defines a family of risk measures by using the concept of *distortion function* as introduced by Greco (1982), Schmeidler (1989) and Yaari (1987); see also Denneberg (1994), Wang, Young & Panjer (1997) and Dhaene *et al.* (2004). A distortion function is a non-decreasing function $g : [0,1] \to [0,1]$ satisfying $g(0) = 0$ and $g(1) = 1$. The distortion risk measure associated with distortion function g is denoted by $\rho_g[\cdot]$ and is defined by

$$\rho_g[X] = -\int_{-\infty}^{0} [1 - g(P[X > x])]\,\mathrm{d}x + \int_{0}^{\infty} g(P[X > x])\,\mathrm{d}x, \tag{6}$$

for any random variable X, provided that the integrals converge. The risk measure $\rho_g[X]$ can be interpreted as a "distorted expectation" of X, evaluated with a "distorted probability measure" in the sense of a Choquet integral; see e.g., Denneberg (1994). As is well-known, concave distortion risk measures (induced by a concave distortion function) are a subclass of the class of coherent risk measures.

Föllmer & Schied (2002) introduce the concept of *convex* risk measures, which satisfy the properties of monotonicity, translation invariance and convexity; see also Frittelli & Rosazza Gianin (2002). The interested reader is referred to Deprez & Gerber (1985) for early work in this direction. Föllmer & Schied (2002) argue that, due to, for example, liquidity reasons, the risk of a position increases in a nonlinear way with its size, hence violating the axioms of subadditivity and positive homogeneity. The class of coherent risk measures can be characterized as the class of convex risk measures that satisfy the positive homogeneity property. As the class of convex risk measures is larger than the class of coherent risk measures, it is sometimes called the class of *weakly coherent* risk measures. Though this paper restricts attention to investigating the desirability of the subadditivity property for solvency capital requirements, the reader may verify that most of the results hold as well for the desirability of the convexity property for solvency capital requirements.

In general, the properties that a risk measure should satisfy depend on the risk preferences in the economic environment under consideration. The sets of axioms discussed above should be regarded as typical (and appealing) sets. From a normative point of view, the "best set of axioms" is however nonexistent, as any normative axiomatic setting is based on a "belief" in the axioms. Different sets of axioms for risk measurement may represent different schools of thought. In this respect, the terminology "coherent" can be somewhat misleading as it may suggest that any risk measure that is not "coherent", but for example, convex only, is always inadequate.

3 The required solvency capital

Consider a portfolio with future loss X. As explained above, the regulator wants the solvency capital requirement related to X to be sufficiently large, to ensure that the shortfall risk is sufficiently small. We suppose that, to reach this goal, the regulator introduces a risk measure for the shortfall risk, which we will denote by φ:

$$\varphi\left[(X-\rho[X])_+\right]. \tag{7}$$

From (7), we see that two different risk measures are involved in the process of setting solvency capital requirements: the risk measure ρ that determines the solvency capital requirement and the risk measure φ that measures the shortfall risk.

We will assume that φ satisfies the following condition:

$$\rho_1[X] \leq \rho_2[X] \Rightarrow \varphi\left[(X-\rho_1[X])_+\right] \geq \varphi\left[(X-\rho_2[X])_+\right], \tag{8}$$

which means that an increase of the solvency capital requirement implies a reduction of the shortfall risk as measured by φ. A sufficient condition for (8) to hold is that φ is monotonic.

Assumption (8) implies that the larger the capital, the better from the viewpoint of minimizing $\varphi\left[(X-\rho[X])_+\right]$. The regulator wants $\varphi\left[(X-\rho[X])_+\right]$ to be sufficiently small. However, holding a capital $\rho[X]$ involves a capital cost $\rho[X]\, i$, where i denotes the required excess return on capital. To avoid imposing an excessive burden on the insurer, the regulator should take this capital cost into account. For a given X and a given solvency capital requirement $\rho[X]$, we consider the cost function $C(X,\ \rho[X])$ given by

$$C(X,\ \rho[X]) = \varphi\left[(X-\rho[X])_+\right] + \rho[X]\ \varepsilon, \qquad 0<\varepsilon<1, \tag{9}$$

which takes into account the shortfall risk and the capital cost. Here, ε can be interpreted as a measure for the extent to which the capital cost is taken into account. The regulatory authority can decide to let ε be company-specific or risk-specific. The optimal capital requirement $\rho[X]$ can now be determined as the smallest amount d that minimizes the cost function $C(X,\, d)$. In the limiting case that $\varepsilon = 0$, the capital cost is not taken into account at all and an optimal solvency capital $\rho[X] = \inf\left\{d \mid \varphi\left[(X-d)_+\right] = 0\right\}$ arises. Here, we use the convention that $\inf\{\phi\} = \infty$.

Increasing the value of ε means that the regulator increases the relative importance of the cost of capital. This will result in a decrease of the optimal capital requirement.

Take as an example $\varphi[X] = E[X]$, where the expectation is taken with respect to the base probability measure P as introduced above (we note that by the arbitrariness of P, the results below remain valid if the expectation is taken with respect to any other probability measure on $\mathcal{F}$). Clearly, this choice of φ satisfies condition (8) and moreover satisfies all the axioms listed in Section 2. In this case, the shortfall risk measure can be interpreted as the net stop-loss premium that has to be paid to reinsure the insolvency risk. We state the following result:

Theorem 1 *The smallest element in the set of minimizers to the cost function $C(X,\, d)$ defined by*

$$C(X,\, d) = E\left[(X-d)_+\right] + d\ \varepsilon, \qquad 0<\varepsilon<1, \tag{10}$$

is given by

$$\rho[X] = Q_{1-\varepsilon}[X]. \tag{11}$$

Proof. Though an analytic proof can readily be obtained by differentiating $C(X,\, d)$ with respect to d, we prefer a geometric proof. Let us first assume that $Q_{1-\varepsilon}[X] \geq 0$. When $d \geq 0$, the cost function $C(X,\, d)$ corresponds with the surface between the distribution function of X and the horizontal line $y = 1$, from d on, together with the surface $d\varepsilon$; see Figure 1. A similar interpretation for $C(X,\, d)$ as a surface holds when $d < 0$. One can easily verify that $C(X,\, d)$ is decreasing in d if $d \leq Q_{1-\varepsilon}[X, d]$ while $C(X,\, d)$ is increasing in d if $d \geq Q_{1-\varepsilon}[X]$. We can conclude that the cost function $C(X,\, d)$ is minimized by choosing $d = Q_{1-\varepsilon}[X]$.
Let us now assume that $Q_{1-\varepsilon}[X] < 0$. A similar geometric reasoning leads to the conclusion that also in this case, the cost function $C(d)$ is minimized by $Q_{1-\varepsilon}[X]$.
Note that the minimum of (10) is uniquely determined, except when $(1-\varepsilon)$ corresponds to a flat part of the distribution function. In the latter case, the minimum is obtained for any x for which $F_X(x) = 1 - \varepsilon$. Determining the capital requirement as the smallest amount for which the cost function in (10) is minimized leads to the solution (11). ■

From the proof of the theorem, we see that for values of $d \geq Q_{1-\varepsilon}[X]$, the marginal increase of the capital cost exceeds the marginal decrease of the expected shortfall. For values of $d \leq Q_{1-\varepsilon}[X]$, the opposite holds.

The set of minimizers of the function $C(X,\, d)$ as defined in (10) is equal to the set of minimizers of the function $\tilde{C}(X,\, d)$ defined by

$$\tilde{C}(X,\, d) = (1-\varepsilon)\ E\left[(X-d)_+\right] + \varepsilon\ E\left[(d-X)_+\right]. \tag{12}$$

This follows from the fact that the cost function $C(X,\, d)$ can also be written as $C(X,\, d) = (1-\varepsilon)\ E\left[(X-d)_+\right] + \varepsilon\ E\left[(d-X)_+\right] + \varepsilon\ E[X]$. Minimizing the function $\tilde{C}(\cdot, d)$ has been considered (in another context) in Ferguson (1967) and Hinderer (1972); see for more details Acerbi & Tasche (2002).[2]

Remark 2 *From (4) it follows that the minimal value of the cost function in (10) can be expressed as*

$$C(X,\ Q_{1-\varepsilon}[X]) = E\left[(X - Q_{1-\varepsilon}[X])_+\right] + Q_{1-\varepsilon}[X]\,\varepsilon = \varepsilon\, TVaR_{1-\varepsilon}[X]. \tag{13}$$

Theorem 1 provides a possible theoretical justification for the use of Value-at-Risk to set solvency capital requirements. Hence, to some extent the theorem supports the current regulatory regime for banking supervision established by the Basel Capital Accord, which has put forward a Value-at-Risk-based capital requirement approach (see Basel Committee (1988, 1996, 2004)).

[2] We thank Dirk Tasche for mentioning this to us.

It is important to emphasize that the Value-at-Risk is not used to "measure risk" here; it (merely) appears as an optimal capital requirement. The risk that we measure and want to keep under control is the shortfall $(X - \rho[X])_+$. This shortfall risk is measured by $E\left[(X - \rho[X])_+\right]$. This approach corresponds to the classical actuarial approach of measuring or comparing risks by determining or comparing their respective stop-loss premiums. Therefore, the well-known problems of Value-at-Risk-based risk management caused by not taking into account the shortfall risk and leading to an impetus to go for gambling portfolios (see among others Basak & Shapiro (2001)), do not apply to our context.

In Theorem 1, we determined the optimal capital requirement $\rho[X]$ by minimizing the cost function $C(X,\ d)$ over all possible values of d. Another way of determining the optimal capital requirement is to minimize the cost function $C(X,\ d)$ over a restricted set of possible values for d. For instance, we could restrict the set of possible capital requirements to the class of concave distortion risk measures that lead to a capital requirement that is at least as large as the optimal capital requirement in the unconstrained problem. This minimization problem is considered in the next theorem.

Theorem 3 *The smallest element in the set of minimizers to the minimization problem*

$$\min_{d \epsilon A} \ C(X,\ d), \tag{14}$$

where the cost function $C(X,\ d)$ is defined by (10) and the set A is defined by

$$A = \{\rho_g[X] \mid g \text{ is a concave distortion function and } \rho_g[X] \geq Q_{1-\varepsilon}[X]\}, \tag{15}$$

is given by

$$\rho[X] = TVaR_{1-\varepsilon}[X]. \tag{16}$$

Proof. It can be proven that the smallest element contained in the set A is given by $\mathrm{TVaR}_{1-\varepsilon}[X]$; see Dhaene *et al.* (2004). Furthermore, from the proof of Theorem 1, it follows that the cost function $C(\cdot, d)$ is non-decreasing if $d \geq Q_{1-\varepsilon}[X]$. This proves the theorem. ∎

The theorem states that if one wants to set the capital requirement such that it belongs to the class of concave distortion risk measures (and hence, is subadditive), such that it is the smallest minimizer of the problem (14) and such that is not smaller than the smallest minimizer of the unconstrained problem, then the optimal capital requirement is given by the Tail-Value-at-Risk at level $1 - \varepsilon$.

4 Diversification and the subadditivity property

In this section, we discuss the subadditivity condition that is often imposed on solvency capital principles. We consider two portfolios with respective future losses X_1 and X_2. We assume that the solvency capital requirement imposed by the regulator in force is represented by the risk measure ρ. We say that the portfolios are merged when they are jointly liable for the shortfall of the aggregate loss $X_1 + X_2$. The solvency capital requirement imposed by the supervisory authority will in this case be equal to $\rho\left[X_1 + X_2\right]$. When each of the portfolios is not liable for the shortfall of the other portfolio, we will say that they are stand-alone portfolios. In this case, the solvency capital requirement for each portfolio is given by $\rho\left[X_j\right]$. Throughout, we assume that the losses X_1 and X_2 remain the same, regardless of whether or not the portfolios are merged, and that only the (legal) liability construction changes. In practice, merging or splitting portfolios may change management, business strategy, cost structure, and so on, and may thus change the losses under consideration.

Let us first consider the case of splitting a merged portfolio into two stand-alone portfolios. This will result in a change in shortfall given by

$$\sum_{j=1}^{2} \left(X_j - \rho\left[X_j\right]\right)_+ - \left(X_1 + X_2 - \rho\left[X_1 + X_2\right]\right)_+ . \tag{17}$$

As mentioned in Dhaene, Goovaerts & Kaas (2003), the following implication holds: if ρ is superadditive, then

$$\left(X_1 + X_2 - \rho\left[X_1 + X_2\right]\right)_+ \leq \sum_{j=1}^{2} \left(X_j - \rho\left[X_j\right]\right)_+ . \tag{18}$$

In particular, we have that

$$\left(X_1 + X_2 - \rho\left[X_1\right] - \rho\left[X_2\right]\right)_+ \leq \sum_{j=1}^{2} \left(X_j - \rho\left[X_j\right]\right)_+ . \tag{19}$$

From (19) we can draw the following conclusion: assume that the solvency capital requirement is additive; in this case, splitting a merged portfolio into two separate entities leads to an increase of the shortfall. Hence, from the regulator's point of view, splitting a merged portfolio leads to a less favorable situation when the solvency capital requirement is additive. The same holds when the solvency capital requirement is superadditive; see (18). Only a risk measure that is "sufficiently subadditive" can guarantee that splitting portfolios will not imply an increase of the shortfall.

Let us now consider the converse case of merging two stand-alone portfolios. Inequality (19) states that the shortfall of the merged portfolio is always smaller than the sum of the shortfalls of the stand-alone portfolios, when the solvency capital requirement is additive. It expresses that, from the viewpoint of the regulatory authority, a merger is desirable in the sense that the shortfall decreases, when the solvency capital requirement is additive. The underlying reason is that within the merged portfolio, the shortfall of one of the entities can be compensated by the gain of the other one, which is the diversification benefit of the merger. This observation can be summarized as: "*a merger decreases the shortfall*". Moreover, only taking into account the criterion of minimizing the shortfall, inequality (19) indicates that the solvency capital of the merged portfolios can to a certain extent be smaller than the sum of the solvency capitals of the two stand-alone portfolios.

The above observations support the belief (of many academics and practitioners) that a solvency capital requirement should be subadditive. Indeed, when splitting a portfolio, the solvency capital requirement should be sufficiently subadditive to prevent an increase of the shortfall risk. When merging two stand-alone portfolios, subadditivity may be allowed to some extent by the regulator, as long as the shortfall risk of the merged portfolio does not become larger than the sum of the shortfalls of the stand-alone portfolios. In axiomatic approaches to capital allocation, the property of subadditivity is often considered as one of the axioms.

Important to notice is that the requirement of subadditivity implies that

$$(X_1 + X_2 - \rho\left[X_1 + X_2\right])_+ \geq (X_1 + X_2 - \rho\left[X_1\right] - \rho\left[X_2\right])_+ , \tag{20}$$

and consequently, for some realizations (x_1, x_2) we may have that

$$(x_1 + x_2 - \rho\left[X_1 + X_2\right])_+ > (x_1 - \rho\left[X_1\right])_+ + (x_2 - \rho\left[X_2\right])_+ . \tag{21}$$

Hence, when applying a subadditive risk measure in a merger, one could end up with a larger shortfall than the sum of the shortfalls of the stand-alone entities. Therefore, the regulatory authority needs to restrict the subadditivity in order to avoid that merging leads to a riskier situation. In the following sections, we will further investigate the problem of how to avoid that a risk measure for setting solvency capital requirements is too subadditive in the sense that a merger leads to an increase of the shortfall risk.

Remark 4 *We emphasize here that the comparison of the shortfall risk between the merged and the stand-alone situation may not be the only concern of the regulator. For instance, it is well-known in practice that merging portfolios typically increases the probability of systemic failure, i.e., the probability of a complete breakdown of the system, because* *removing (legal) fire walls increases the risk of financial contagion; see e.g., Danielsson et al. (2005). Therefore, though subadditivity may be desirable to reflect the diversification benefit of a merger, in view of e.g., the systemic failure probability, the desirability of the subadditivity property for solvency capital requirements is questionable.*

It is important to note that inequality (19) does not necessarily express that merging is advantageous for the owners of the business related to the portfolios (i.e., the shareholders). Evaluating whether a merger is advantageous for them can be done by comparing the returns on capital for the two situations. Let X_j denote the loss (claim payments minus premiums) over the reference period related to portfolio j and let K_j denote its available capital, $j = 1, 2$. If the loss X_j is smaller than the capital K_j, the capital at the end of the reference period will be given by $K_j - X_j$, whereas in case the loss X_j exceeds K_j, the business unit related to this portfolio gets ruined and the end-of-the-year capital equals 0. Hence, for portfolio j the end-of-the-year capital is given by $(K_j - X_j)_+$. Since

$$(K_1 + K_2 - X_1 - X_2)_+ \leq \sum_{j=1}^{2} (K_j - X_j)_+ \,, \tag{22}$$

for maximizing the end-of-the-period capital, it is advantageous to keep the two portfolios separated. This situation may be preferred from the shareholders' point of view, essentially because in this case fire walls are built in, ensuring that the ruin of one portfolio will not contaminate the other one. Notice that the optimal strategy from the owners' point of view is now just the opposite of the optimal strategy from the regulator's point of view. Inequality (22) justifies the well-known advice "*don't put all your eggs in one basket*". If the shareholders have a capital $K_1 + K_2$ at their disposal, if the riskiness of the business is given by (X_1, X_2), and if their goal is to maximize the return on capital, then splitting the risks over two stand-alone entities is always to be preferred.

To conclude: when the regulator talks about diversification, the decrease in shortfall caused by merging is meant. When the shareholders talk about diversification, the increase in investment return caused by building in fire walls is meant.

5 Avoiding that a merger increases the shortfall risk

As we observed in the previous section, any theory that postulates that risk measures for solvency capital requirements are subadditive should constraint this subadditivity; this to avoid that merging, which leads to a lower aggregate solvency capital requirement, increases the shortfall risk. In this section, we will investigate a number of requirements that could be imposed by the regulator in addition to the subadditivity requirement, in order to ensure that the merger will indeed lead to a less risky situation.

A first additional condition could be as follows:
For any random couple (X_1, X_2), the solvency capital requirement ρ has to satisfy the condition

$$(X_1 + X_2 - \rho[X_1 + X_2])_+ \leq \sum_{j=1}^{2} (X_j - \rho[X_j])_+ . \tag{23}$$

When imposing this condition, the regulator requires that the shortfall after a merger of two portfolios with losses X_1 and X_2 is never larger than the sum of the shortfalls of the stand-alone portfolios. We state the following theorem:

Theorem 5 *Consider for a given solvency capital requirement ρ a random couple (X_1, X_2) for which*

$$\Pr\left[X_1 > \rho\left[X_1\right], X_2 > \rho\left[X_2\right]\right] > 0 \tag{24}$$

holds. If ρ satisfies condition (23) *for this random couple, then one has that*

$$\rho\left[X_1 + X_2\right] \geq \rho\left[X_1\right] + \rho\left[X_2\right]. \tag{25}$$

Proof. Consider the random couple (X_1, X_2) that satisfies condition (24). Let us assume that $\rho\left[X_1 + X_2\right] \leq \rho\left[X_1\right] + \rho\left[X_2\right]$. Then, from condition (23), we find that

$$\begin{aligned} & E\left[(X_1 + X_2 - \rho[X_1 + X_2])_+ \mid X_1 > \rho\left[X_1\right], X_2 > \rho\left[X_2\right]\right] \\ \leq & \sum_{j=1}^{2} E\left[(X_j - \rho[X_j])_+ \mid X_1 > \rho\left[X_1\right], X_2 > \rho\left[X_2\right]\right]. \end{aligned} \tag{26}$$

From this inequality and the assumption that $\rho\left[X_1 + X_2\right] \leq \rho\left[X_1\right] + \rho\left[X_2\right]$, one immediately finds that $\rho\left[X_1 + X_2\right] = \rho\left[X_1\right] + \rho\left[X_2\right]$ must hold. This proves the stated result. ■

An immediate consequence of the theorem is that any capital requirement ρ that is subadditive and that satisfies condition (23) must necessarily be additive for all random couples (X_1, X_2) for which (24) holds true. Hence, any such capital requirement is "almost" additive. Only random couples (X_1, X_2) that are sufficiently negatively dependent, in the sense that $\Pr\left[X_1 > \rho\left[X_1\right], X_2 > \rho\left[X_2\right]\right] = 0$, may lead to a capital requirement for the merged portfolio that is strictly smaller than the sum of the requirements for the stand-alone portfolios.

The theorem illustrates the fact that the subadditivity axiom and condition (23) are in fact not compatible. If the regulatory authority requires that a merge of portfolios will never increase the shortfall, then it cannot propose a subadditive risk measure.

Note that from the proof of Theorem 5, we see that condition (23) in that theorem can be weakened to condition (26).

Let us now weaken condition (23). We impose that the solvency capital requirement ρ is such that the *expected* shortfall after a merger does not exceed the sum of the *expected* shortfalls of the stand-alone portfolios. Hence, we will impose that ρ satisfies the following additional condition for all random couples (X_1, X_2):

$$E\left[(X_1 + X_2 - \rho[X_1 + X_2])_+\right] \leq \sum_{j=1}^{2} E\left[(X_j - \rho\left[X_j\right])_+\right]. \tag{27}$$

The subadditivity condition together with condition (27) ensures that the capital will be decreased in case of a merger, but only to such an extent that on average the situation does not become riskier.

In the following theorem we prove that in case of bivariate normal random variables, condition (27) is fulfilled for a broad class of risk measures ρ.

Theorem 6 *For any translation invariant and positively homogeneous risk measure ρ and any bivariate normally distributed random couple (X_1, X_2), we have that condition (27) is fulfilled.*

Proof. Assume that (X_1, X_2) is bivariate normal with $var\left[X_j\right] = \sigma_j^2$ and $var\left[X_1 + X_2\right] = \sigma^2$.
Let Z be a standard normally distributed random variable. Then we immediately find

$$E\left[(X_j - \rho\left[X_j\right])_+\right] = \sigma_j E\left[(Z - \rho\left[Z\right])_+\right]$$

and

$$E\left[(X_1 + X_2 - \rho\left[X_1 + X_2\right])_+\right] = \sigma E\left[(Z - \rho\left[Z\right])_+\right].$$

From

$$\sigma \leq \sigma_1 + \sigma_2$$

we find the stated result. ■

The theorem states that under normality assumptions a translation invariant and positively homogeneous risk measure can never be too subadditive. This result is independent of whether or not ρ is subadditive. In particular, it holds for the Value-at-Risk (which, as is well-known, is subadditive under normality assumptions when the probability level $p \geq 0.5$). The theorem also implies that, when assuming normality, any translation invariant and positively homogeneous risk measure will always lead to an increase of the expected shortfall in case of splitting risks. One could say that under the conditions of the theorem, "the hunger for subadditivity can never be satisfied".

The theorem can easily be generalized to the rich class of bivariate elliptical distributions, which is the class of random couples (X_1, X_2) of which the characteristic function can be expressed as

$$E\left[\exp\left(i\left(t_1 X_1 + t_2 X_2\right)\right)\right] = \exp\left(i\mathbf{t}^T\mu\right) \cdot \phi\left(\mathbf{t}^T\mathbf{\Sigma}\mathbf{t}\right), \qquad \mathbf{t} = (t_1\ t_2)^T, \tag{28}$$

for some scalar function ϕ, a 2-dimensional vector μ and where $\mathbf{\Sigma}$ is of the form $\mathbf{\Sigma} = \mathbf{A}\mathbf{A}^T$ for some $2\times m$ matrix $\mathbf{A}$. The function ϕ is called the *characteristic generator* of (X_1, X_2). Notice that the characteristic generator of the bivariate normal distribution is given by $\phi(u) = \exp(-u/2)$. A standard reference for the theory of elliptical distributions is Fang, Kotz & Ng (1987). For applications of elliptical distributions in insurance and finance, see Landsman & Valdez (2002).

Theorem 6 could give the impression that under very general conditions, the requirement (27) holds true. However, this is not the case, not even for Tail-Value-at-Risk, which is undoubtedly the best-known *subadditive* risk measure for setting solvency capital requirements. In the following example we illustrate that Tail-Value-at-Risk does not in general satisfy condition (27).

Example 7 *Suppose that X_1 is uniformly distributed on the unit interval $(0,1)$. Let X_2 be the random variable defined by*

$$X_2 = \begin{cases} 0.9U & \text{if } 0 < X_1 \leq 0.9, \\ X_1 & \text{if } 0.9 < X_1 < 1, \end{cases}$$

where U is uniformly distributed on $(0,1)$ and independent of X_1.
For the uniformly distributed random variables X_j we have that

$$TVaR_p[X_j] = \frac{1+p}{2},$$

and

$$E[X_j - TVaR_p[X_j]]_+ = \frac{(1-p)^2}{8}.$$

For $p = 0.85$, we find that the Tail-Value-at-Risk and the expected shortfall are given by

$$TVaR_{0.85}[X_j] = 0.925$$

and

$$E[X_j - TVaR_{0.85}[X_j]]_+ = 0.0028125.$$

Consider now the sum $S = X_1 + X_2$.
For $0 < s < 2$, we find

$$\begin{aligned} F_S(s) &= \Pr\left[S \leq s,\ 0 < X_1 \leq 0.9\right] + \Pr\left[S \leq s,\ X_1 > 0.9\right] \\ &= \int_0^{0.9} \Pr\left[U \leq \frac{s - x_1}{0.9}\right] dx_1 + \Pr\left[0.9 < X_1 \leq 0.5s\right] \end{aligned}$$

Hence, the distribution function of S is given by

$$F_S(s) = \begin{cases} \frac{s^2}{1.8} & : 0 < s \leq 0.9, \\ -\frac{s^2}{1.8} + 2s - 0.9 & : 0.9 < s \leq 1.8, \\ \frac{s}{2} & : 1.8 < s < 2. \end{cases}$$

For $0.9 < d \leq 1.8$ we have that

$$\begin{aligned} E\left[(S-d)_+\right] &= \int_d^{1.8} [1 - (2s - \frac{s^2}{1.8} - 0.9)]ds + \int_{1.8}^{2} (1 - \frac{s}{2})ds \\ &= -\frac{d^3}{5.4} + d^2 - 1.9d + 1.27. \end{aligned}$$

For $p = 0.85$, we find that $Q_{0.85}[S] = 1.5$. This implies that

$$\begin{aligned} TVaR_{0.85}[S] &= Q_{0.85}[S] + \frac{1}{0.15} E\left[(S - Q_{0.85}[S])_+\right] \\ &= 1.8. \end{aligned}$$

Note that $TVaR_{0.85}[S]$ is strictly smaller than $TVaR_{0.85}[X_1] + TVaR_{0.85}[X_2]$.
The expected shortfall of S is given by

$$E[(S - TVaR_{0.85}[S])_+] = 0.01.$$

One can verify that the expected shortfall of S is strictly larger than the sum of the expected shortfalls of the X_j's:

$$E\left[(S - TVaR_{0.85}[S])_+\right] > \sum_{j=1}^{2} E\left[(X_j - TVaR_{0.85}[X_j])_+\right] = 0.006.$$

The example above illustrates the fact that subadditive risk measures, in particular Tail-Value-at-Risk, can be too subadditive, in the sense that the expected shortfall of a merged portfolio is larger than the sum of the expected shortfalls of the two stand-alone portfolios.

6 The regulator's condition

In the previous section we considered conditions that could be imposed in addition to the subadditivity axiom in order to ensure that a merger does not lead to a riskier situation in terms of shortfalls. We found some particular results, but we did not yet find a general satisfying solution. In this section, we will investigate a different approach.

On the one hand, the regulator wants the expected shortfall to be as small as possible, which means a preference for a large solvency capital requirement. On the other hand, the regulator does not want to decrease the expected shortfall at any price, imposing a large burden on the insurance industry.

Taking into account the above considerations, we propose the following requirement that a risk measure ρ for determining the solvency capital required for a risky business should satisfy:

For any random couple (X_1, X_2) and a given number $0 < \varepsilon < 1$, the solvency capital requirement ρ has to satisfy the condition

$$
\begin{aligned}
& E\left[(X_1 + X_2 - \rho[X_1 + X_2])_+\right] + \rho\left[X_1 + X_2\right]\varepsilon \\
\leq\ & \sum_{j=1}^{2} \left\{E\left[(X_j - \rho\left[X_j\right])_+\right] + \rho\left[X_j\right]\varepsilon\right\}.
\end{aligned} \tag{29}
$$

The condition (29) can be interpreted as a compromise between the requirement of "subadditivity" and the requirement of "not too subadditive". We will call it the *regulator's condition*. Here, ε can be equal to the required excess return on capital, but it could also be a number smaller than the required excess return on capital, depending on the extent to which the regulator is willing to take this cost into account.

Theorem 6 above can be adjusted to the following formulation:

Theorem 8 *For any translation invariant, positively homogeneous and subadditive risk measure ρ and any bivariate normal random couple (X_1, X_2), the regulator's condition (29) is fulfilled for any $0 < \varepsilon < 1$.*

The result of Theorem 8 can easily be extended to the case of elliptical random couples. Hence, for elliptical random couples, any coherent risk measure satisfies the regulator's condition.

Let us now consider the case of general random loss variables. We state the following theorem:

Theorem 9 *The capital requirement $\rho[X] = Q_{1-\varepsilon}[X]$ fulfills the regulator's condition (29). Also, any subadditive capital requirement $\rho[X] \geq Q_{1-\varepsilon}[X]$ fulfills the regulator's condition.*

Proof. The regulator's condition (29) can be expressed in terms of the cost function $C(X, d)$ introduced in Theorem 1:

$$C(X_1 + X_2, \rho[X_1 + X_2]) \leq C(X_1, \rho[X_1]) + C(X_2, \rho[X_2]).$$

The proof for $Q_{1-\varepsilon}$ follows immediately from (13) and the subadditivity of Tail-Value-at-Risk.

Let us now consider a subadditive capital requirement $\rho \geq Q_{1-\varepsilon}$. From

$$Q_{1-\varepsilon}(X_1 + X_2) \leq \rho(X_1 + X_2) \leq \rho(X_1) + \rho(X_2)$$

and the fact that $C(X_1 + X_2, d)$ is increasing in d if $d \geq Q_{1-\varepsilon}[X_1 + X_2]$, we find

$$C(X_1 + X_2, \rho[X_1 + X_2]) \leq C(X_1 + X_2, \rho[X_1] + \rho[X_2]).$$

Furthermore, from (19) we find

$$C(X_1 + X_2, \rho[X_1] + \rho[X_2]) \leq C(X_1, \rho[X_1]) + C(X_2, \rho[X_2]),$$

which proves the stated result. ■

Assume that the regulator wants to set the capital requirement ρ as the one that fulfills the regulator's condition (29) and also makes the cost function $E\left[(X - \rho[X])_+\right] + \rho[X]\varepsilon$ minimal for every X. Combining Theorems 1 and 9, we find that the solution to this problem is given by $Q_{1-\varepsilon}$, i.e., the Value-at-Risk of probability level $1 - \varepsilon$.

Let us now assume that the regulator wants to use a subadditive risk measure that fulfills the regulator's condition (29). From Theorem 9, we have that any TVaR_p with $p \geq 1 - \varepsilon$ belongs to this class. Furthermore, $\text{TVaR}_{1-\varepsilon}$ is the smallest concave distortion risk measure that is larger than $Q_{1-\varepsilon}$ (see also Theorem 3 of this paper) and fulfills the regulator's condition (29). Notice that the level of the optimal Value-at-Risk or Tail-Value-at-Risk under consideration depends explicitly on ε, i.e., on the extent to which the capital cost is taken into account.

Because of the arbitrariness of the base probability measure P, most of the results in this paper remain valid when the expectation is calculated with respect to any other probability measure on $\mathcal{F}$. For instance, Theorem 9 remains valid if $Q_{1-\varepsilon}$ is calculated with respect to a distorted probability measure. A version of the minimization problem

(9) with φ being a distortion risk measure, is considered in Dhaene, Goovaerts & Kaas (2003), Laeven & Goovaerts (2004) and Goovaerts, Van den Borre & Laeven (2005).[3]

7 Conclusion

This paper considers the problem of determining appropriate solvency capital requirements to be set by a regulatory authority. We showed that the Value-at-Risk arises as the "most efficient" solvency capital requirement in an intuitive minimization problem with a cost function that balances the expected shortfall and the capital cost.

Next, we discussed the condition of subadditivity that is often imposed on solvency capital principles. As is well-known, the Value-at-Risk does not in general satisfy the subadditivity property (although it does for various particular cases). We showed that subadditivity "to some extent" is justified by the diversification benefit obtained when merging portfolios. We also demonstrated how an "unconstrained" subadditivity can lead to the undesirable situation that a merger leads to an increase of the shortfall risk, and we introduced the *regulator's condition* as a possible remedy to this problem. Replacing the subadditivity condition by the regulator's condition leads to the Value-at-Risk as the optimal solvency capital requirement. Imposing the regulator's condition to the class of concave distortion risk measures (of which the elements, in contrast to the Value-at-Risk, satisfy the subadditivity property), leads to the Tail-Value-at-Risk as the optimal

[3] To illustrate this, consider as an example the proportional hazard (PH) distortion function given by $g(x) = x^{1/\alpha}$, $\alpha \geq 1$, advocated by Wang (1996) and Wang, Young & Panjer (1997). Here, the value of the parameter α determines the degree of risk aversion: the larger the value of α, the larger the risk aversion, with $\alpha = 1$ corresponding to the non-distorted (base) case. Applying Theorem 3.1 of Laeven & Goovaerts (2004), we find that the solution to problem (9) when using for φ the distortion risk measure induced by a PH distortion function, is indeed given by $Q_{1-\varepsilon}$, when calculated with respect to a PH distorted probability measure. Equivalently, this solution can be regarded as a Value-at-Risk of probability level $1 - \varepsilon^{\alpha}$, when calculated with respect to the base probability measure P.

Suppose that the regulatory authority sets ε equal to 4%. Then, the table below displays the probability level of the Value-at-Risk when calculated with respect to the base probability measure P, for various values of the parameter α.

α	Probability level w.r.t. P
1.0	96.00%
1.2	97.90%
1.4	98.90%
1.6	99.42%
1.8	99.70%
2.0	99.84%

solvency capital requirement. In both cases, the probability level of the (Tail-) Value-at-Risk depends explicitly on the extent to which the capital cost is taken into account.

An issue that is left undiscussed in the paper, but that is relevant in practice when determining appropriate solvency capital requirements, is the practical tractability of the risk measure. Recall that the Tail-Value-at-Risk of probability level p is equal to the average of the Value-at-Risks of level q, with $q \geq p$. Because the standard error of the estimator of the Value-at-Risk typically increases when one goes further in the tail of the loss distribution, it is clear that adequately and robustly estimating a Tail-Value-at-Risk is more involved than estimating a Value-at-Risk of the same probability level. This problem will be particularly relevant in the case of heavy-tailed loss variables. Although this consideration should perhaps not play a role in a discussion on *optimal* solvency capital requirements, it clearly is a main concern in practice.

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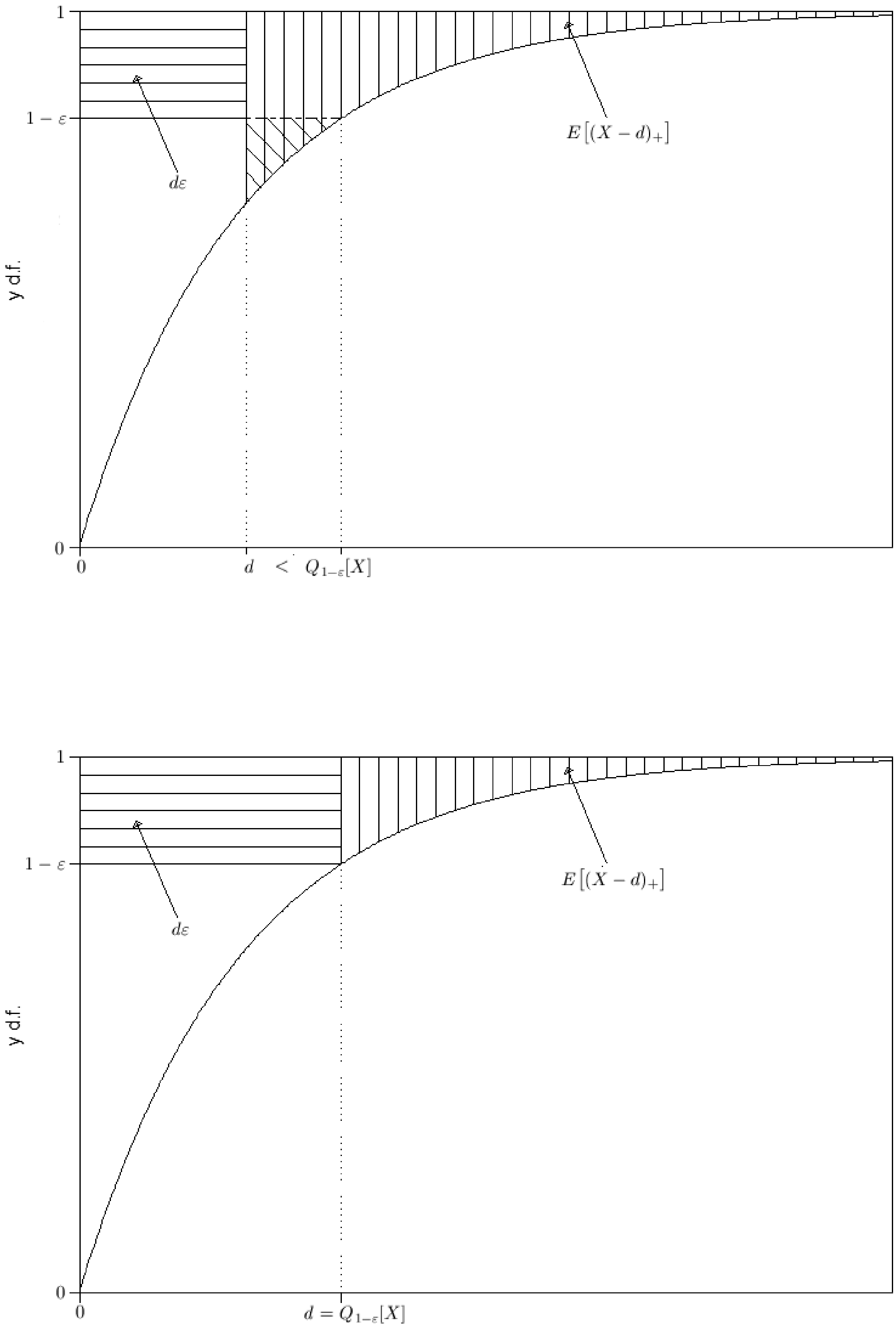
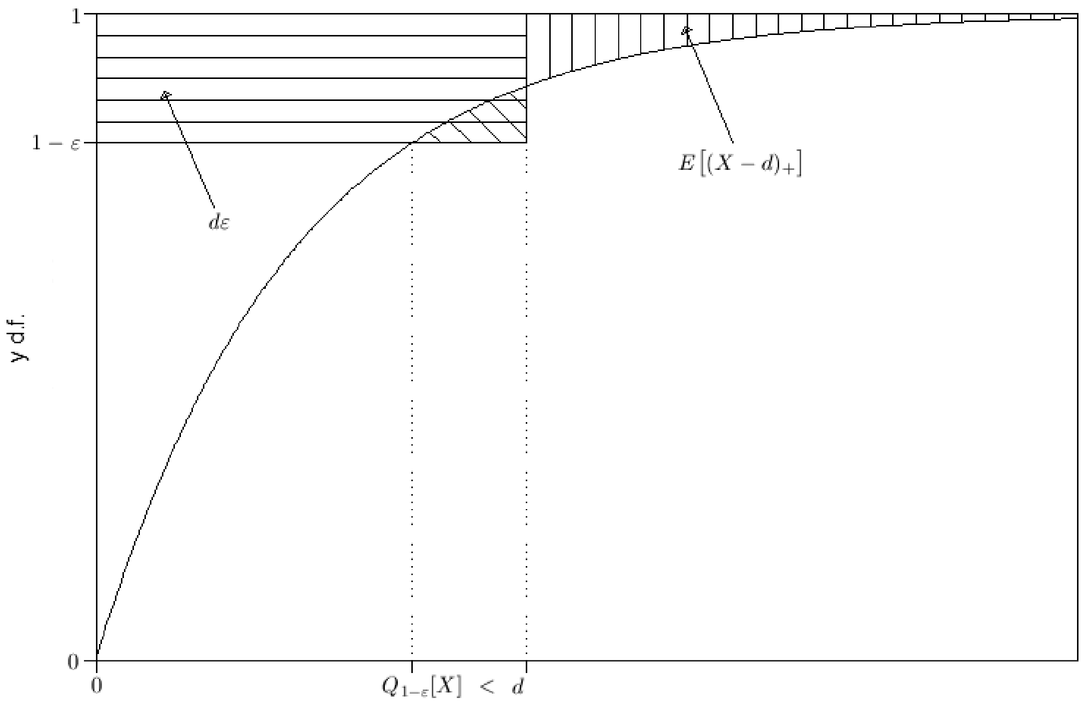
Figure 1: Geometric Proof of Theorem 1





y d.f.
1
$1-\varepsilon$
0
0
$d\varepsilon$
$E\left[(X-d)_+\right]$
$Q_{1-\varepsilon}[X] \; < \; d$

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